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Wronskian-type solutions for the vector k-constrained KP hierarchy

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Abstract. Motivated by a relation of the 1-constrained Kadomtsev–Petviashvili (KP) hierarchy with the 2-component KP hierarchy, the tau-functions of the vector k-constrained KP hierarchy are constructed by using an analogue of the Baker–Akhiezer $(m+1)$-point function. These tau-functions are expressed in terms of Wronskian-type determinants.

1. Introduction

In recent years, a series of papers have been devoted to the study of a class of integrable systems which are constrained from the Kadomtsev–Petviashvili (KP) hierarchy [1–6]; as in [5] we call this class of integrable systems the *vector k-constrained KP hierarchy*. For arbitrary given positive integers k, m, the vector k-constrained KP hierarchy can be expressed as [3–5]

$$L^k_{t_n} = [B_n, L^k] \tag{1.1a}$$

$$q_{i,t_n} = B_n q_i \tag{1.1b}$$

$$r_{i,t_n} = -B_n^* r_i \tag{1.1c}$$

$$i = 1, 2, \ldots, m \qquad n \geqslant 2 \tag{1.1d}$$

where the micro-differential operator L is defined as

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots \tag{1.2}$$

which satisfies the following condition

$$L^k = B_k + \sum_{i=1}^{m} q_i \partial^{-1} r_i \tag{1.3}$$

u_i's, q_i's and r_i's are functions of the variable $t = (t_1, t_2, \ldots)$, $\partial = \partial/\partial x$ with $x = t_1$, B_n is the differential part of the micro-differential operator L^n, B_n^* is the operator adjoints to B_n and $\partial^{-1} r_i$ is defined as

$$\partial^{-1} r_i = r_i \partial^{-1} - r_{i,x}\partial^{-2} + r_{i,xx}\partial^{-3} - \cdots. \tag{1.4}$$

The hierarchy of equations in (1.1) can be represented in terms of the dynamical variables $u_2, u_3, \ldots, u_k$ and $q_i, r_i (i = 1, 2, \ldots, m)$. When $m = 1$, we call the hierarchy of equations in (1.1) the *k-constrained KP hierarchy*. These integrable systems are proved to possess

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the Lax pair representations, the bi-Hamiltonian structures and the bilinear representations [3–5, 7], and they are also equivalent to some integrable systems which are closely related to the study of W-algebras, multi-matrix models [8–11] and topological field theory [12].

In [5] we studied the solutions of the vector k-constrained KP hierarchy (1.1) by employing the bilinear method, we showed how to obtain their rational and soliton-like solutions starting from the solutions of the KP hierarchy; these rational and soliton-like solutions can be expressed by vertex operators. In [13], we studied the Wronskian structure of the solutions of the k-constrained KP hierarchy, we proved that the Yajima–Oikawa equation which is the first flow in the 2-constrained KP hierarchy has solutions in generalized double Wronskian form, then based on this and some other facts we conjectured that the general k-constrained KP hierarchy also possess solutions in generalized double Wronskian form, and we also conjectured the form of these solutions.

The purpose of the present paper is to construct the Wronskian-type solutions for the vector k-constrained KP hierarchy (1.1), and as an aside to prove our conjecture given in [13]. We shall employ the method presented in [14], where soliton-like solutions for some integrable systems were constructed; these integrable systems are equivalent to the first flows of the vector k-constrained KP hierarchy (1.1) with $k = 1$–4. This method arises from the algebraic-geometric method in soliton theory [15, 16]; it starts from the construction of an analogue of the Baker–Akhiezer function for the KP hierarchy (the Baker–Akhiezer 1-point function) by solving certain algebraic linear system and then imposes on the constructed function some self-consistency conditions to obtain the solutions for the relevant integrable systems. To construct the Wronskian-type solutions for the vector k-constrained KP hierarchy (1.1), we shall start from the construction of an analogue of the Baker–Akhiezer $(m + 1)$-point function instead of starting from the construction of the analogue of the Baker–Akhiezer 1-point function. The advantage of our construction lies in the fact that we can obtain solutions of the whole hierarchy (1.1) in a straightforward way and can express these solutions in a simple form by using some Wronskian-type determinants. The motivation of our construction comes from the relation of the 1-constrained KP hierarchy (i.e. the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, see [3]) with the 2-component KP hierarchy. This relation enables us to construct the solutions of the 1-constrained KP hierarchy from the solutions of the 2-component KP hierarchy in a much more straightforward way than to construct the solutions of the 1-constrained KP hierarchy by imposing some constraints on solutions of the KP hierarchy, as was done in [5, 14].

In section 2 we explain our motivation by the construction of the double Wronskian solutions for the first flow of the 1-constrained KP hierarchy, in section 3 we construct the solutions for the vector k-constrained KP hierarchy (1.1), in section 4 we construct the tau-functions of the hierarchy (1.1) and show that they can be expressed in Wronskian-type determinants and in section 5 we give some concluding remarks.

2. Double Wronskian solutions for the first flow of the 1-constrained KP hierarchy as a motivation

In this section, we construct the double Wronskian solutions of the first flow of the 1-constrained KP hierarchy by using its relation with the 2-component KP hierarchy, and thus give a hint of the construction of the Wronskian-type solutions for the general vector k-constrained KP hierarchy. For the convenience of our further construction of Wronskian-type solutions we will state this well known relation [17, 18] in the language of the method given in [14].

Let us assume that the functions $\psi_1^{(i)}(x,y,\lambda)$, $\psi_2^{(i)}(x,y,\zeta)$ have the following forms:

$$\psi_1^{(i)}(x,y,\lambda) = \left(\delta_1^i + \sum_{j=1}^{M} u_{1j}^{(i)}(x,y)\lambda^{-j}\right)e^{x_1\lambda + y_1\lambda^2} \tag{2.1a}$$

$$\psi_2^{(i)}(x,y,\zeta) = \left(\delta_2^i + \sum_{j=1}^{N} u_{2j}^{(i)}(x,y)\zeta^{-j}\right)e^{x_2\zeta + y_2\zeta^2} \tag{2.1b}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, δ_j^i is the Kronecker-delta function and λ, ζ are two parameters, and $i = 1, 2$. These functions give rise an analogue of the Baker–Akhiezer 2-point function [15, 16]. To specify the coefficients $u_{lj}^{(i)}$'s, let us impose the following linear conditions on the functions $\psi_j^{(i)}$:

$$a_l\psi_1^{(i)}(x,y,\lambda_l) + b_l\psi_2^{(i)}(x,y,\zeta_l) = 0 \qquad l = 1,2,\ldots,M+N \qquad i = 1,2 \tag{2.2}$$

where $a_l, b_l, \lambda_l, \zeta_l$ are some given constants. Let us define

$$U^{(i)} = (u_{11}^{(i)},\ldots,u_{1M}^{(i)},u_{21}^{(i)},\ldots,u_{2N}^{(i)})^T \tag{2.3}$$

and rewrite the linear conditions in (2,2) as

$$\mathbf{A}U^{(i)} = \mathbf{F}^{(i)} \qquad i = 1,2 \tag{2.4}$$

where $\mathbf{A}$ and $\mathbf{F}^{(i)}$'s are $(M+N)\times(M+N)$ and $(M+N)\times 1$ matrices, respectively. We assume that the matrix $\mathbf{A}$ is not identically degenerate; since the determinant of A is a smooth function of x, y, we can assume in what follows that $\mathbf{A}$ is non-degenerate in some open domain and (x, y) belongs to this domain.

Lemma 2.1. Denote $u_{11}^{(i)} = r_{i1}, u_{21}^{(i)} = r_{i2}$, then the functions $\psi_1^{(i)}, \psi_2^{(i)}$ satisfy the following system of linear equations:

$$\psi_{j,x_2}^{(1)} = r_{12}\psi_j^{(2)} \tag{2.5a}$$

$$\psi_{j,x_1}^{(2)} = r_{21}\psi_j^{(1)} \tag{2.5b}$$

$$\psi_{j,y_1}^{(1)} = \psi_{j,x_1x_1}^{(1)} - 2r_{11,x_1}\psi_j^{(1)} \tag{2.6a}$$

$$\psi_{j,y_2}^{(1)} = \psi_{j,x_2x_2}^{(1)} - 2r_{12,x_2}\psi_j^{(2)} \tag{2.6b}$$

$$\psi_{j,y_1}^{(2)} = \psi_{j,x_1x_1}^{(2)} - 2r_{21,x_1}\psi_j^{(1)} \tag{2.6c}$$

$$\psi_{j,y_2}^{(2)} = \psi_{j,x_2x_2}^{(2)} - 2r_{22,x_2}\psi_j^{(2)}. \tag{2.6d}$$

Proof. From the form of $\psi_j^{(i)}$ we see that $\phi_j := \psi_{j,x_2}^{(1)} - r_{12}\psi_j^{(2)}$ has the form $(\sum_{j=1}^{M} v_{1j}(x,y)\lambda^{-j})e^{x_1\lambda+x_2\lambda^2}$ for $j = 1$ and has the form $(\sum_{j=1}^{N} v_{2j}(x,y)\lambda^{-j})e^{y_1\zeta+y_2\zeta^2}$ for $j = 2$, and from (2.2) we see that ϕ_j's satisfy

$$a_l\phi_1(x,y,\lambda_l) + b_l\phi_2(x,y,\zeta_l) = 0 \qquad 1 \leqslant l \leqslant M+N. \tag{2.7}$$

If we define

$$V = (v_{11},\ldots,v_{1M},v_{21},\ldots,v_{2N})^T \tag{2.8}$$

then conditions in (2.7) can be written as

$$\mathbf{A}V = 0 \tag{2.9}$$

and thus from the non-degeneracy of $\mathbf{A}$ we obtain that $\phi_j = 0$, which proves the identity (2.5a); the identities (2.5b) and (2.6a)–(2.6d) can be proved similarly. The lemma is proved. □

From the equalities $\psi^{(2)}_{2,x_1} = r_{21}\psi^{(1)}_2$, $\psi^{(1)}_{1,x_2} = r_{12}\psi^{(2)}_1$ and the form of $\psi^{(i)}_j$'s we can easily see that $r_{22,x_1} = r_{11,x_2} = r_{12}r_{21}$. This fact and the compatibility conditions of the linear equations in (2.5) and (2.6) lead to

Theorem 2.1. The functions r_{ij} satisfy the following system of equations

$$r_{12,y_1} = r_{12,x_1x_1} - 2r_{12}r_{11,x_1} \tag{2.10a}$$
$$r_{12,y_2} = -r_{12,x_2x_2} + 2r_{12}r_{22,x_2} \tag{2.10b}$$
$$r_{21,y_1} = -r_{21,x_1x_1} + 2r_{21}r_{11,x_1} \tag{2.10c}$$
$$r_{21,y_2} = r_{21,x_2x_2} - 2r_{21}r_{22,x_2} \tag{2.10d}$$
$$r_{11,x_2} = r_{12}r_{21} \tag{2.10e}$$
$$r_{22,x_1} = r_{12}r_{21}. \tag{2.10f}$$

The equations in (2.10) are the simpliest non-trivial flows of the 2-component KP hierarchy [17, 18]. If we introduce the following new variables:

$$t_1 = x_1 - x_2 \qquad \tilde{t}_1 = x_1 + x_2 \qquad t_2 = y_1 - y_2 \qquad \tilde{t}_2 = y_1 + y_2. \tag{2.11}$$

Then the equations in (2.10) lead to the Davey–Stewartson system [17], and if we require that r_{ij}'s are independent of $\tilde{t}_i (i = 1, 2)$, then the equations in (2.10) lead to

$$\partial_{t_2} r_{12} = \partial^2_{t_1} r_{12} + 2(r_{12}r_{21})r_{12} \tag{2.12a}$$
$$\partial_{t_2} r_{21} = -\partial^2_{t_1} r_{21} - 2(r_{12}r_{21})r_{21} \tag{2.12b}$$

which is the first non-trivial flow in the 1-constrained KP hierarchy [3].

Lemma 2.2. Let us assume $\lambda_l = \zeta_l$ for $1 \leqslant l \leqslant M + N$ in (2.2), then r_{ij}'s constructed from (2.1)–(2.4) are independent of $\tilde{t}_1$ and $\tilde{t}_2$.

Proof. Since $\lambda_l = \zeta_l$, we can eliminate the factor $\mathrm{e}^{\frac{1}{2}\tilde{t}_1\lambda_l + \frac{1}{2}\tilde{t}_2\lambda_l^2}$ from the linear systems in (2.2), and obtain linear systems which do not depend on $\tilde{t}_1$ and $\tilde{t}_2$. Thus the r_{ij}'s which are constructed from (2.1)–(2.4) are independent of $\tilde{t}_1$ and $\tilde{t}_2$. We have proved the lemma. $\square$

Let us now define the following two vectors,

$$\varphi = (a_1\lambda_1^{-M}\mathrm{e}^{\frac{1}{2}t_1\lambda_1 + \frac{1}{2}t_2\lambda_1^2}, \ldots, a_{M+N}\lambda_{M+N}^{-M}\mathrm{e}^{\frac{1}{2}t_1\lambda_{M+N} + \frac{1}{2}t_2\lambda_{M+N}^2})^T \tag{2.13a}$$
$$\phi = (b_1\lambda_1^{-N}\mathrm{e}^{-\frac{1}{2}t_1\lambda_1 - \frac{1}{2}t_2\lambda_1^2}, \ldots, b_{M+N}\lambda_{M+N}^{-N}\mathrm{e}^{-\frac{1}{2}t_1\lambda_{M+N} - \frac{1}{2}t_2\lambda_{M+N}^2})^T \tag{2.13b}$$

and we also denote $s = \frac{1}{2}t_1$, then from lemma 2.2 and the linear conditions in (2.2) we have

Theorem 2.2. The system of equations in (2.12) has the following double Wronskian solution [19, 20]:

$$r_{12} = \frac{\rho}{\tau} \qquad r_{21} = \frac{\sigma}{\tau} \tag{2.14}$$

where

$$\rho = -\det(\varphi, \partial_s\varphi, \ldots, \partial_s^M\varphi, \phi, \partial_s\phi, \ldots, \partial_s^{N-2}\phi) \tag{2.15a}$$
$$\sigma = -\det(\varphi, \partial_s\varphi, \ldots, \partial_s^{M-2}\varphi, \phi, \partial_s\phi, \ldots, \partial_s^{N}\phi) \tag{2.15b}$$
$$\tau = \det(\varphi, \partial_s\varphi, \ldots, \partial_s^{M-1}\varphi, \phi, \partial_s\phi, \ldots, \partial_s^{N-1}\phi). \tag{2.15c}$$

Remark 2.1. Under the condition $\lambda_l = \zeta_l (1 \leqslant l \leqslant M+N)$, we know that the functions $\psi_j^{(i)}(x, y, \lambda)\mathrm{e}^{-x_2\lambda - y_2\lambda^2}$, $i, j = 1, 2$, do not depend on $\tilde{t}_1$ and $\tilde{t}_2$, so we can denote them as $\varphi_j^{(i)}(t, \lambda)$, where $t = (t_1, t_2)$. Then instead of condition (2.2) we have

$$a_l\varphi_1^{(i)}(t, \lambda_l) + b_l\varphi_2^{(i)}(t, \lambda_l) = 0 \qquad i = 1, 2, 1 \leqslant l \leqslant M+N. \tag{2.16}$$

We also define

$$\tilde{\tau}(t) = c(t)\tau(t) \qquad \tilde{\rho}(t) = c(t)\rho(t) \qquad \tilde{\sigma}(t) = c(t)\sigma(t) \tag{2.17}$$

where $c(t) = \exp[(t_1/2)\sum_{l=1}^{M+N}\lambda_l + (t_2/2)\sum_{l=1}^{M+N}\lambda_l^2]$. Then the functions $\varphi_j^{(i)}(t, \lambda)$, $\tilde{\tau}(t), \tilde{\rho}(t)$ and $\tilde{\sigma}(t)$ can be written as functions of $\mathrm{e}^{t_1\lambda_l + t_2\lambda_l^2}$, $1 \leqslant l \leqslant M+N$. We now replace the term $\mathrm{e}^{t_1\lambda_l + t_2\lambda_l^2}$ in these functions by the term $\exp[\sum_{n=1}^{\infty} t_n\lambda_l^n]$ and denote $t = (t_1, t_2, \ldots, t_n, \ldots)$. Then we can express $\varphi_1^{(i)}$ in the following form (see theorem 4.1, later):

$$\varphi_1^{(1)}(t, \lambda) = \frac{\tilde{\tau}(t - \epsilon(\lambda))}{\tilde{\tau}(t)} \exp\left[\sum_{n=1}^{\infty} t_n\lambda^n\right] \qquad \varphi_1^{(2)}(t, \lambda) = \frac{\tilde{\sigma}(t - \epsilon(\lambda))}{\lambda\tilde{\tau}(t)} \exp\left[\sum_{n=1}^{\infty} t_n\lambda^n\right] \tag{2.18}$$

where $\epsilon(\lambda) = (1/\lambda, 1/2\lambda^2, 1/3\lambda^3, \ldots)$. The above formulae show us that $\varphi_1^{(i)}(t, \lambda)$, $i = 1, 2$, are the wavefunctions of the 1-constrained KP hierarchy defined in [5]; this fact gives us a motivation for our further construction of the Wronskian-type solutions of the vector k-constrained KP hierarchy.

3. Solutions for the vector k-constrained KP hierarchy (1.1)

Let us define the following functions:

$$\varphi_1^{(i)}(t, \lambda) = \left(\delta_1^i + \sum_{\nu=1}^{M_1} u_{1\nu}^{(i)}(t)\lambda^{-\nu}\right)\mathrm{e}^{\xi(t,\lambda)} \tag{3.1a}$$

$$\varphi_j^{(i)}(t, \lambda) = \delta_j^i + \sum_{\nu=1}^{M_j} u_{j\nu}^{(i)}(t)\lambda^{-k\nu} \tag{3.1b}$$

$$2 \leqslant j \leqslant m+1 \qquad 1 \leqslant i \leqslant m+1$$

where $M_j, 1 \leqslant j \leqslant m+1$ are some positive integers, λ is a parameter, $t = (t_1, t_2, \ldots, t_n, \ldots)$, and

$$\xi(t, \lambda) = \sum_{l=1}^{\infty} t_l\lambda^l. \tag{3.2}$$

We specify the coefficients $u_{j\nu}^{(i)}$'s by imposing the following linear conditions on $\varphi_j^{(i)}$:

$$\sum_{\nu=1}^{m+1} a_{l\nu}\varphi_\nu^{(i)}(t, \lambda_l) = 0 \qquad 1 \leqslant l \leqslant M,\ 1 \leqslant i \leqslant m+1 \tag{3.3}$$

where $M = \sum_{\mu=1}^{m+1} M_\mu$ and $a_{l\nu}$'s are some given constants. If we denote

$$U^{(i)} = (u_{11}^{(i)}, \ldots, u_{1M_1}^{(i)}, u_{21}^{(i)}, \ldots, u_{2M_2}^{(i)}, \ldots, u_{(m+1)1}^{(i)}, \ldots, u_{(m+1)M_{(m+1)}}^{(i)})^T \tag{3.4}$$

then the linear conditions in (3.3) can be written as

$$\mathbf{A}U^{(i)} = \mathbf{F}^{(i)} \tag{3.5}$$

where $\mathbf{A}$ and $\mathbf{F}^{(i)}$'s are $M \times M$ and $M \times 1$ matrices, respectively. As in section 2 we assume in what follows that the matrix $\mathbf{A}$ is non-degenerate in a certain open domain and we assume that t belongs to this domain, then the linear systems of equations in (3.5) uniquely determine the functions $\varphi_j^{(i)}$'s, which give an analogue of the Baker–Akhiezer $(m+1)$-point function (see remark 2.1).

We define now the following operators:

$$P = 1 + \sum_{\nu=1}^{M_1} u_{1\nu}^{(1)}(t)\partial^{-\nu} \tag{3.6a}$$

$$L = P\partial P^{-1} \tag{3.6b}$$

where $\partial = \partial/\partial t_1$. Let us also denote

$$q_i(t) = u_{(i+1)1}^{(1)} \qquad r_i(t) = u_{11}^{(i+1)} \qquad 1 \leqslant i \leqslant m. \tag{3.7}$$

Then we have

Lemma 3.1. The functions $\varphi_j^{(i)}$'s defined by (3.1)–(3.5) satisfy the following system of linear equations:

$$B_k\varphi_j^{(1)} + \sum_{\mu=1}^{m} q_\mu \varphi_j^{(\mu+1)} = \lambda^k \varphi_j^{(1)} \tag{3.8a}$$

$$\varphi_{j,t_1}^{(\nu+1)} = r_\nu \varphi_j^{(1)} \tag{3.8b}$$

$$\varphi_{j,t_n}^{(1)} = B_n \varphi_j^{(1)} \tag{3.8c}$$

$$\varphi_{j,t_n}^{(\nu+1)} = A_n^{(\nu)} \varphi_j^{(1)} \tag{3.8d}$$

$$1 \leqslant \nu \leqslant m \qquad n \geqslant 2$$

where $B_n = (L^n)_+$ is the differential part of the micro-differential operator L^n, $A_n^{(\nu)}$ is a $(n-1)$th order differential operator which has the form

$$A_n^{(\nu)} = \sum_{l=0}^{n-1} f_{nl}^{(\nu)} \partial^l \tag{3.9}$$

and is uniquely defined by the following relations:

$$\partial A_n^{(\nu)} = r_\nu B_n - (B_n^* r_\nu) \qquad n \geqslant 2 \tag{3.10}$$

here B_n^* is the operator adjoints to B_n.

Proof. Let us denote

$$\Phi_j = B_k \varphi_j^{(1)}(t,\lambda) - \lambda^k \varphi_j^{(1)}(t,\lambda) + \sum_{\mu=1}^{m} q_\mu \varphi_j^{(\mu+1)}(t,\lambda). \tag{3.11}$$

From the definition of $\varphi_1^{(1)}(t,\lambda)$ and P we have $\varphi_1^{(1)} = P\mathrm{e}^{\xi(t,\lambda)}$, so from (3.6$b$) we obtain

$$L^k \varphi_1^{(1)} = \lambda^k \varphi_1^{(1)}. \tag{3.12}$$

By using (3.12) and (3.1) we see that Φ_1 has the form

$$\left(\sum_{\nu=1}^{M_1} w_{1\nu}(t)\lambda^{-\nu} \right) \mathrm{e}^{\xi(t,\lambda)}. \tag{3.13}$$

On the other hand, from (3.1) we see that Φ_j for $2 \leqslant j \leqslant m+1$ has the form

$$\sum_{\nu=1}^{M_j} w_{j\nu}\lambda^{-k\nu}. \tag{3.14}$$

The linear conditions in (3.3) lead to

$$\sum_{\nu=1}^{m+1} a_{l\nu}\Phi_\nu(t,\lambda_l) = 0 \qquad 1 \leqslant l \leqslant M \tag{3.15}$$

and this turns out to be a homogenous linear system for the unknowns $w_{l\nu}$'s due to (3.13) and (3.14). From the non-degeneracy of the coefficient matrix $\mathbf{A}$, we obtain that $\Phi_j = 0$.

In a similar way, we can show that the equalities (3.8*b*) and (3.8*c*) hold and we can show that there exist operators $A_n^{(\nu)}$'s of the form (3.9) such that (3.8*d*) holds. To specify the operators $A_n^{(\nu)}$'s, we use the compatibility conditions of (3.8*b*) and (3.8*d*), which lead to

$$\partial A_n^{(\nu)} = r_\nu B_n + r_{\nu,t_n}. \tag{3.16}$$

We denote B_n by $\sum_{\mu=0}^{n} b_\mu \partial^\mu$, then substitute (3.9) into the equality (3.16) so we obtain

$$f_{n(n-1)}^{(\nu)} = r_\nu b_n \qquad f_{nl}^{(\nu)} = r_\nu b_{l+1} - \partial_{t_1} f_{n(l+1)}^{(\nu)} \qquad 0 \leqslant l \leqslant n-2 \tag{3.17a}$$

$$r_{\nu,t_n} = -B_n^* r_\nu \tag{3.17b}$$

thus we have proved the lemma. □

Relations in (3.8) coincide with the Lax representation of the vector k-constrained KP hierarchy [5], from which we have the following theorem:

Theorem 3.1. The operator L and the functions $q_i(t), r_i(t), 1 \leqslant i \leqslant m$, which are constructed in this section from (3.1)–(3.7) gave a solution of the vector k-constrained KP hierarchy (1.1).

Proof. From (3.8*a*)–(3.8*c*) and (3.12) we see that L satisfies (1.1*a*) and (1.3), and (3.17*b*) shows that $r_i(t)$ satisfies (1.1*c*). To prove that q_i's satisfy the equations in (1.1*b*), let us take the derivative of both sides of (3.8*a*) with respect to t_n. By using (3.8*c*) we have

$$\partial_{t_n}(B_k\varphi_j^{(1)}) + \sum_{\mu=1}^{m} q_{\mu,t_n}\varphi_j^{(\mu+1)} + \sum_{\mu=1}^{m} q_\mu \varphi_{j,t_n}^{(\mu+1)} = \lambda^k B_n \varphi_j^{(1)} \tag{3.18}$$

where we assume that $2 \leqslant j \leqslant m+1$; then by using (3.1*b*) and equating the coefficients of λ^0 in both sides of (3.18) we obtain $q_{j-1,t_n} = B_n q_{j-1}, 2 \leqslant j \leqslant m+1$. The theorem is proved. □

In the next section, we show that the solutions of the vector k-constrained KP hierarchy constructed in this section can be expressed in terms of tau-functions which are some Wronskian-type determinants.

4. Wronskian-type tau-functions of the vector k-constrained KP hierarchy (1.1)

Let us keep in mind the notations of section 3, and define

$$\phi_1 = (a_{11}\lambda_1^{-M_1}\mathrm{e}^{\frac{1}{2}\xi(t,\lambda_1)}, a_{21}\lambda_2^{-M_1}\mathrm{e}^{\frac{1}{2}\xi(t,\lambda_2)}, \ldots, a_{M1}\lambda_M^{-M_1}\mathrm{e}^{\frac{1}{2}\xi(t,\lambda_M)})^T \tag{4.1a}$$

$$\phi_j = (a_{1j}\lambda_1^{-kM_j}\mathrm{e}^{-\frac{1}{2}\xi(t,\lambda_1)}, a_{2j}\lambda_2^{-kM_j}\mathrm{e}^{-\frac{1}{2}\xi(t,\lambda_2)}, \ldots, a_{Mj}\lambda_M^{-kM_j}\mathrm{e}^{-\frac{1}{2}\xi(t,\lambda_M)})^T \tag{4.1b}$$

$2 \leqslant j \leqslant m+1$.

We denote $s = t_1/2$ as in section 2, and define the following two matrices:

$$\mathbf{R} = (\phi_1, \partial_s\phi_1, \ldots, \partial_s^{M_1}\phi_1, \phi_2, \partial_s^k\phi_2, \ldots, \partial_s^{k(M_2-1)}\phi_2, \ldots, \phi_{m+1}, \partial_s^k\phi_{m+1}, \ldots, \partial_s^{k(M_{m+1}-1)}\phi_{m+1}) \tag{4.2a}$$

$$\mathbf{Q} = (\phi_1, \partial_s\phi_1, \ldots, \partial_s^{M_1-2}\phi_1, \phi_2, \partial_s^k\phi_2, \ldots, \partial_s^{k(M_2-1)}\phi_2, \ldots, \phi_{m+1}, \partial_s^k\phi_{m+1}, \ldots, \partial_s^{k(M_{m+1}-1)}\phi_{m+1}). \tag{4.2b}$$

We also define

$$\tau(t) = \mathrm{e}^{\frac{1}{2}\sum_{l=1}^{M}\xi(t,\lambda_l)}\det(\mathbf{R}(n_0+1)) \tag{4.3a}$$

$$\rho_j(t) = (-1)^{n_j-M_1+k(M_{j+1}-1)}\mathrm{e}^{\frac{1}{2}\sum_{l=1}^{M}\xi(t,\lambda_l)}\det(\mathbf{R}(n_j+1)) \tag{4.3b}$$

$$\sigma_j(t) = (-1)^{n_j-M_1+kM_{j+1}+1}\mathrm{e}^{\frac{1}{2}\sum_{l=1}^{M}\xi(t,\lambda_l)}\det(\mathbf{Q}(n_j-1, \partial_s^{kM_{j+1}}\phi_{j+1}) \tag{4.3c}$$

$$1 \leqslant j \leqslant m \qquad n_i = \sum_{l=1}^{i+1} M_l \qquad 0 \leqslant i \leqslant m$$

where $\mathbf{R}(n_j+1)$ is a matrix obtained from the matrix $\mathbf{R}$ by removing its (n_j+1)th column, and $\mathbf{Q}(n_j-1, \partial_s^{kM_{j+1}}\phi_{j+1})$ is a matrix obtained from $\mathbf{Q}$ by adding to the matrix $\mathbf{Q}$ the column $\partial_s^{kM_{j+1}}\phi_{j+1}$ behind its (n_j-1)th column $\partial_s^{k(M_{j+1}-1)}\phi_{j+1}$.

Theorem 4.1. The functions $q_i(t), r_i(t)$ for $i = 1, 2, \ldots, m$ and $\varphi_1^{(j)}$ for $j = 1, 2, \ldots, m+1$ have the following expression:

$$q_i(t) = \frac{\rho_i(t)}{\tau(t)} \qquad r_i(t) = \frac{\sigma_i(t)}{\tau(t)} \tag{4.4}$$

$$\varphi_1^{(1)}(t,\lambda) = \frac{\tau(t-\epsilon(\lambda))}{\tau(t)}\mathrm{e}^{\xi(t,\lambda)} \tag{4.5a}$$

$$\varphi_1^{(j)}(t,\lambda) = \frac{\sigma_{j-1}(t-\epsilon(\lambda))}{\lambda\tau(t)}\mathrm{e}^{\xi(t,\lambda)} \tag{4.5b}$$

$$1 \leqslant i \leqslant m \qquad 2 \leqslant j \leqslant m+1$$

where $\epsilon(\lambda) = (1/\lambda, 1/2\lambda^2, 3/\lambda^3, \ldots)$.

Proof. The relations in (4.4) are evident from (3.1)–(3.7) and the definitions given in (4.1)–(4.3). The relations in (4.5) can be proved by using the fact that [18]

$$\mathrm{e}^{\frac{1}{2}\xi(t,\lambda_l)}\partial_s^j(a_{l1}\lambda_l^{-M_1}\mathrm{e}^{\frac{1}{2}\xi(t,\lambda_l)}) - \lambda^{-1}\mathrm{e}^{\frac{1}{2}\xi(t,\lambda_l)}\partial_s^{j+1}(a_{l1}\lambda_l^{-M_1}\mathrm{e}^{\frac{1}{2}\xi(t,\lambda_l)}) = \mathrm{e}^{\frac{1}{2}\xi(t-\epsilon(\lambda),\lambda_l)}\partial_s^j(a_{l1}\lambda_l^{-M_1}\mathrm{e}^{\frac{1}{2}\xi(t-\epsilon(\lambda),\lambda_l)}). \tag{4.6}$$

The theorem is proved. □

From (4.4), (4.5) and [5] we see that $\tau(t), \rho_i(t), \sigma_i(t), i = 1, 2, \ldots, m$, satisfy the bilinear equations of the vector k-constrained KP hierarchy given in [5], and thus they are the tau-functions of the vector k-constrained KP hierarchy. From (4.5) we also see that the operator P defined in (3.6a) can be expressed by the function $\tau(t)$ as follows:

$$P = \sum_{\nu=0}^{M_1} \frac{p_\nu(-\tilde{\partial})\tau(t)}{\tau(t)}\partial^{-\nu} \tag{4.7}$$

where $p_\nu(t)$'s are the Schur polynomials defined by

$$\mathrm{e}^{\xi(t,\lambda)} = \sum_{\nu=0}^{\infty} p_\nu(t)\lambda^\nu \tag{4.8}$$

and $\tilde{\partial} = (\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \ldots)$. So the operator L is also determined by the function $\tau(t)$, this fact together with (4.4) implies that the tau-functions $\tau(t), \rho_i(t), \sigma_i(t), i = 1, 2, \ldots, m$, characterize a solution of the vector k-constrained KP hierarchy (1.1).

When $m = 1$, the tau-functions have the following form:

$$\tau(t) = c_1(t)\det(\phi_1, \partial_s\phi_1, \ldots, \partial_s^{M_1-1}, \phi_2, \partial_s^k\phi_2, \ldots, \partial_s^{k(M_2-1)}\phi_2) \tag{4.9a}$$

$$\rho_1(t) = c_2(t)\det(\phi_1, \partial_s\phi_1, \ldots, \partial_s^{M_1}\phi_1, \phi_2, \partial_s^k\phi_2, \ldots, \partial_s^{k(M_2-2)}\phi_2) \tag{4.9b}$$

$$\sigma_1(t) = c_3(t)\det(\phi_1, \partial_s\phi_1, \ldots, \partial_s^{M_1-2}\phi_1, \phi_2, \partial_s^k\phi_2, \ldots, \partial_s^{kM_2}\phi_2) \tag{4.9c}$$

where $c_1(t) = \exp[\frac{1}{2}\sum_{l=1}^{M}\xi(t, \lambda_l)]$, $c_2(t) = (-1)^{(k+1)M_2-k}c_1(t)$, $c_3(t) = (-1)^{(k+1)M_2+1}c_1(t)$.

From the bilinear equations of the k-constrained KP hierarchy [5] we see that if $\tau(t)$ and $\rho_1(t)$ and $\sigma_1(t)$ are its tau-functions, then we can define another set of tau-functions for the k-constrained KP hierarchy as follows:

$$\hat{\tau}(t) = \varepsilon_1\tau(-t) \qquad \hat{\rho}_1(t) = \varepsilon_2\sigma_1(-t) \qquad \hat{\sigma}_1 = \varepsilon_2\rho_1(-t) \tag{4.10}$$

where $\varepsilon_i = \pm 1$. It is then not hard to see that the solution of the k-constrained KP hierarchy obtained from the tau-functions in (4.9) is equivalent to the solution constructed from the tau-functions which were given in the conjecture of [13], thus we also proved this conjecture.

5. Concluding remarks

The fact that the 1-constrained KP hierarchy can be reduced from the 2-component KP hierarchy enables us to construct the double Wronskian solutions of the 1-constrained KP hierarchy starting from an analogue of the Baker–Akhiezer 2-point functions. This construction in turn motivated us to construct the Wronskian-type solutions of the vector k-constrained KP hierarchy starting from an analogue of the Baker–Akhiezier $(m+1)$-point function. We remark here that the reduction from the 2-component KP hierarchy to the 1-constrained KP hierarchy is much more straightforward than the reduction from the KP hierarchy to the 1-constrained KP hierarchy; it only needs us to require that the dynamical variables of the 2-component KP hierarchy do not depend on some independent variables, which is similar to the reduction from the KP hierarchy to the Korteweg–de Vries (KdV) hierarchy. We finally remark that our construction of the Wronskian-type solutions of the vector k-constrained KP hierarchy suggests that there may exist a certain analogue of the $(2+1)$-dimensional $(m+1)$-component KP hierarchy from which the vector k-constrained KP hierarchy can be straightforwardly reduced, and it also suggests a relatively convenient way to construct the algebraic–geometric solutions of the vector k-constrained KP hierarchy by using a certain analogue of the Baker–Akhiezer $(m+1)$-point function. These subjects will be discussed in further publications.

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